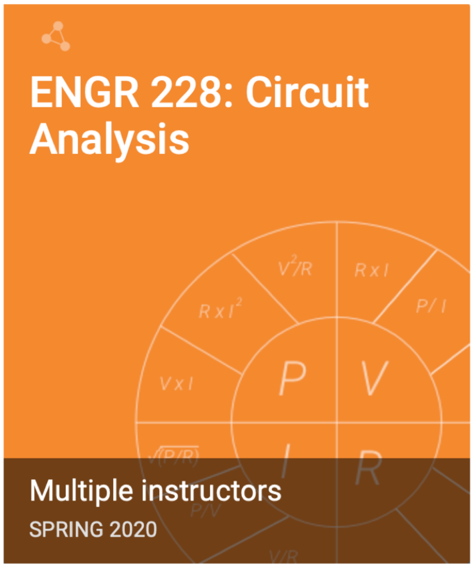


Chapter 7.1,2

Sinusoids and Complex Math

Engr228 - Circuit Analysis

Spring 2020

Dr Curtis Nelson

Sections 7.1,2 Objective

- Review sinusoidal representation and complex math.

Properties of a Sinusoidal Waveform

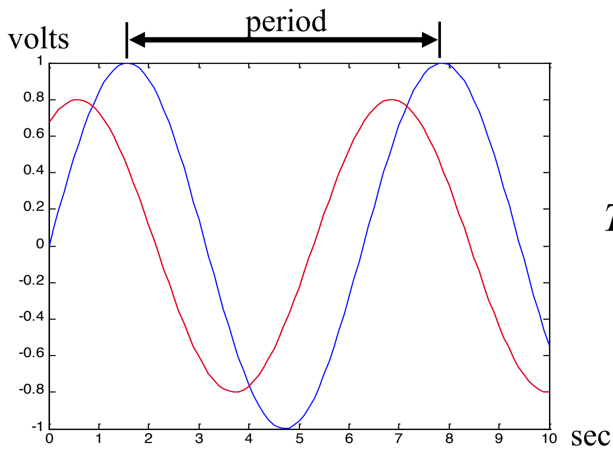
The general form of sinusoidal wave is

$$v(t) = V_m \sin(\omega t + \theta)$$

where:

- V_m is the amplitude in *peak* voltage;
- ω is the angular frequency in radian/second, also $2\pi f$;
- *q* is the phase shift in degrees or radians.

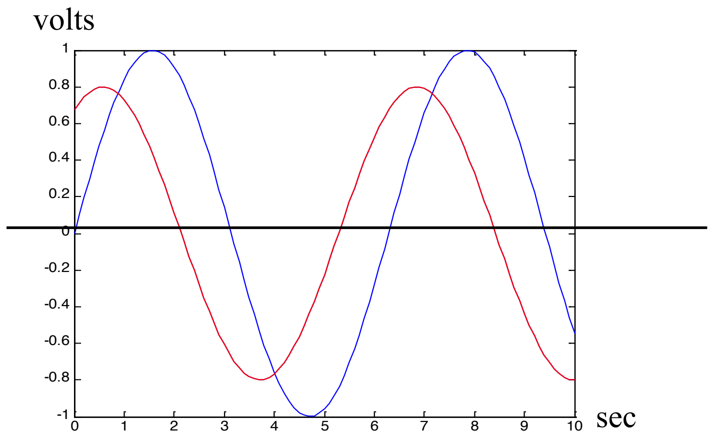
Frequency Review



$$T = \frac{1}{f}$$

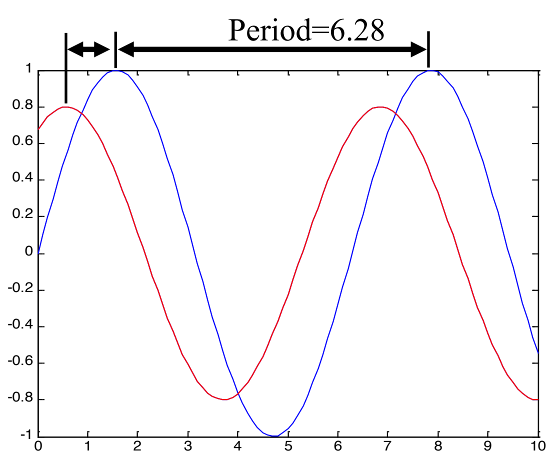
Period ≈ 6.28 seconds, Frequency = 0.1592 Hz

Amplitude Review



Peak: Blue 1 volt, Red 0.8 volts
Peak-to-Peak: Blue 2 volts, Red 1.6 volts
Average: 0 volts

Phase Shift Review

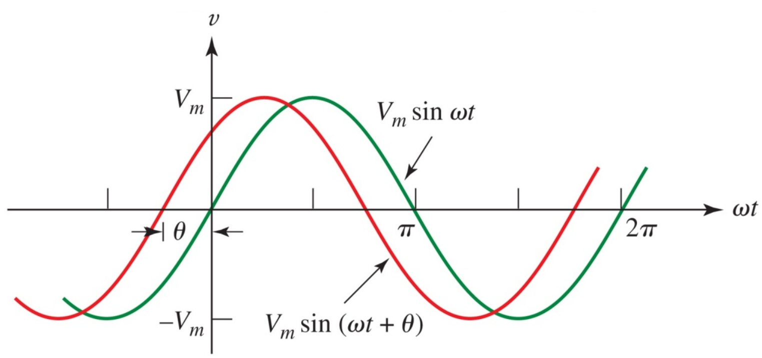


$$y_{blue} = \sin(t)$$
$$y_{red} = 0.8\sin(t+1)$$

Red leads Blue by 57.3 degrees (1 radian) $\phi = \frac{1}{6.28} \times 360^{\circ} = 57.3^{\circ}$

More on Phase

- The red wave [$V_M sin(\omega t + \theta)$] ***leads*** the wave in green by θ;
- The green wave [$V_M sin(\omega t)$] ***lags*** the wave in red by θ;
- The units of θ and ωt must be consistent.



Basic AC Circuit Components

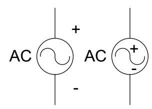
- AC Voltage and Current Sources **(active components)**
- Resistors (R)
- Inductors (L)
- Capacitors (C)

(Resistors, Inductors, Capacitors: **(passive components)**)

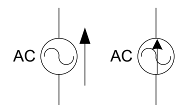
- Inductors and capacitors have limited energy storage capability.

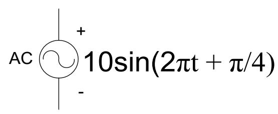
AC Voltage and Current Sources

Voltage Sources



Current Sources





Amplitude = $10V_{peak}$

$\omega = 2\pi$ so F = 1Hz

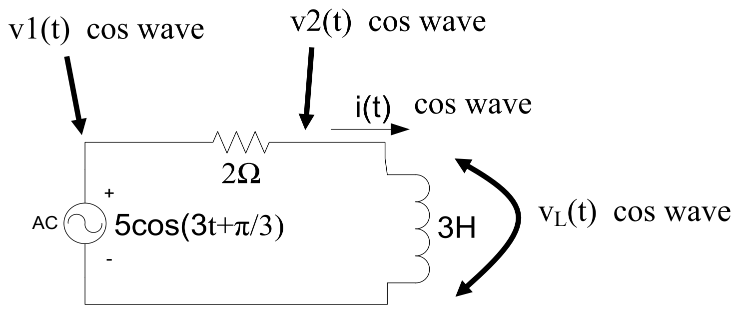
Phase shift = 45°

Sinusoidal Steady State (SSS) Analysis

- SSS is important for circuits containing capacitors and inductors because these elements provide little value in circuits with only DC sources;
- Sinusoidal means that source excitations have the form $V_S cos(\omega t + \theta)$ or $V_S sin(\omega t + \theta)$;
- Since $V_S sin(\omega t + \theta)$ can be written as $V_S cos(\omega t + \theta - \pi/2)$, we will use $V_S cos(\omega t + \theta)$ as the general form for our source excitation;
- Steady state means that all transient behavior in the circuit has decayed to zero.

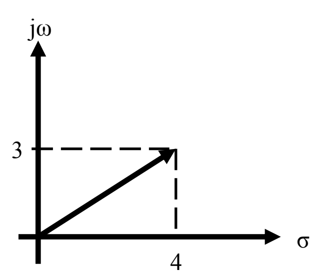
Sinusoidal Steady State Response

The SSS response of a circuit to a sinusoidal input is also a sinusoidal signal ***with the same frequency*** but with possibly different amplitude and phase shift.



Review of Complex Numbers

- Complex numbers can be viewed as vectors where the X-axis represents the real part and the Y-axis represents the imaginary part.
- There are two common ways to represent complex numbers:
  - Rectangular form: $4 + j3$
  - Polar form: $5 \angle 37^{o}$



Complex Number Forms

Rectangular form: $a + jb$

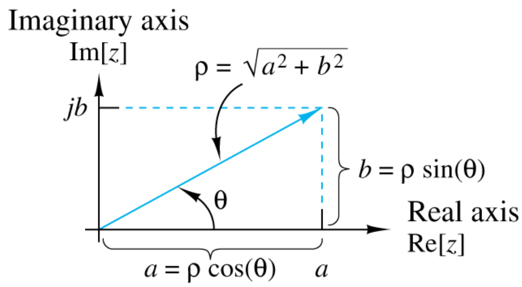
Polar form: $\rho \angle \theta$

$$\rho = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$a = \rho \cos\theta$$

$$b = \rho \sin\theta$$



Complex Math – Rectangular Form

p = a + *j*b **q = c + *j*d**

- Addition and subtraction

x = p + q = (a + c) + *j*(b + d)

y = p – q = (a - c) + *j*(b - d)

- Example

p = 3 + *j*4 **q = 1 - *j*2**

x = p + q = (3 + 1) + *j*(4 - 2) = 4 + *j*2

y = p – q = (3 – 1) + *j*(4 – (-2)) = 2 + *j*6

Complex Math – Rectangular Form

p = a + *j*b **q = c + *j*d**

- Multiplication (easier in polar form)

x = p × q = ac + *j*ad + *j*bc + j^2bd = (ac – bd) + *j*(ad + bc)

- Example

p = 3 + *j*4 **q = 1 - *j*2**

x = p × q = [(3)(1) - (4)(-2)] + *j*[(3)(-2) + (4)(1)]
= 11 – *j*2

Complex Math – Rectangular Form

p = a + *j*b **q = c + *j*d**

- Division (easier in polar form)

$$x = \frac{p}{q} = \frac{a+jb}{c+jd} = \left(\frac{(a+jb)(c-jd)}{(c+jd)(c-jd)}\right) = \left(\frac{(ac+bd)+j(bc-ad)}{c^2+d^2}\right)$$

- Example

p = 3 + *j*4 **q = 1 - *j*2**

$$x = \frac{p}{q} = \frac{((3)(1)+(4)(-2))+j((4)(1)-(3)(-2))}{1^2+(-2)^2} = \frac{-5+j10}{5} = -1+j2$$

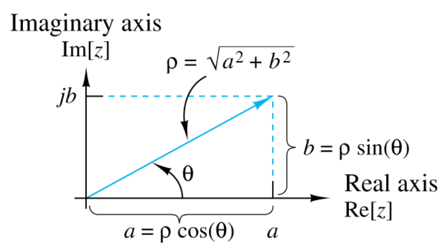
Euler's Identity

- Euler's identity states that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$
- A complex number can then be written as:

$$r = a + jb = \rho\cos(\theta) + j\rho\sin(\theta) = \rho[\cos(\theta) + j\sin(\theta)] = \rho e^{j\theta}$$

- Using shorthand notation, we write this as:

$$\rho e^{j\theta} \equiv \rho \angle \theta$$



Complex Math – Polar Form

$$x = a + jb = \rho e^{j\theta} = \rho\angle\theta \qquad \rho = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$p = m_1 e^{j(\theta_1)} \qquad q = m_2 e^{j(\theta_2)}$$

- Addition and subtraction - too hard in polar so convert to rectangular coordinates.
- Multiplication

$$z = p \times q = m_1 m_2 e^{j(\theta_1 + \theta_2)}$$

- Example

$$p = 6e^{j\left(\frac{\pi}{6}\right)} \qquad q = 2e^{j\left(\frac{\pi}{2}\right)} \qquad z = p \times q = (6)(2)e^{j\left(\frac{\pi}{6} + \frac{\pi}{2}\right)} = 12e^{j\left(\frac{2\pi}{3}\right)}$$

$$p = 6\angle 30° \qquad q = 2\angle 90° \qquad z = p \times q = 12\angle 120°$$

Complex Math – Polar Form

$x = a + jb = \rho e^{j\theta} = \rho\angle\theta \qquad \rho = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$

$p = m_1 e^{j(\theta_1)} \qquad q = m_2 e^{j(\theta_2)}$

- Division

$$z = p \div q = \frac{m_1}{m_2} e^{j(\theta_1 - \theta_2)}$$

- Example

$$p = 6e^{j\left(\frac{\pi}{6}\right)} \qquad q = 2e^{j\left(\frac{\pi}{2}\right)}$$

$$z = p \div q = \frac{6}{2} e^{j\left(\frac{\pi}{6} - \frac{\pi}{2}\right)} = 3e^{j\left(-\frac{\pi}{3}\right)} = 3\angle -60^\circ$$

More on Sinusoids

- Suppose you connect a function generator to any circuit containing resistors, inductors, and capacitors. If the function generator is set to produce a sinusoidal waveform, then ***every*** voltage drop and ***every*** current in the circuit will also be a sinusoid of the ***same*** frequency. Only the amplitudes and phase angles will (may) change.
- The same thing is ***not*** necessarily true for waveforms of other shapes like triangle or square waveforms.
- Fortunately, it turns out that sinusoids are not only the easiest waveforms to work with mathematically, they're also the most useful and occur quite frequently in real-world applications.

Phasors

- A ***phasor*** is a vector that represents an AC electrical quantity such as a voltage waveform or a current waveform;
- The phasor's length represents the peak value of the voltage or current;
- The phasor's angle represents the phase angle of the voltage or current;
- Phasors are used to represent the relationship between two or more waveforms with the same frequency.

Sections 7.1,2 Summary

- Reviewed sinusoid representation and complex math.